Slowly growing graded Lie algebras and complex structures

Dmitry Millionshchikov (Lomonosov Moscow State University)

Following Zelmanov and Shalev we call $\mathbb{N}$-graded Lie algebras $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ with $\dim \mathfrak{g}_i \leq 2$, narrow. In particular, these algebras have slow linear growth. Narrow algebras were studied by Vergne (filiform Lie algebras) long before Zelmanov and Shalev. Carnot algebras are $\mathbb{N}$-graded Lie algebras satisfying $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, i \geq 1$.

We study integrable almost complex structures $J : \mathfrak{g} \to \mathfrak{g}, J^2 = -Id$ on real nilpotent Lie algebras. The integrability means vanishing of the Nijenhuis tensor

$$[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0, \ \forall X, Y \in \mathfrak{g}.$$

It follows from the integrability condition that J defines a left-invariant complex structure on the corresponding nilmanifold G/Γ ($\mathfrak{g}$ is the tangent Lie algebra for the simply connected nilpotent Lie group G). Goze and Remme (2022) proved that filiform Lie algebras do not admit integrable complex stuctures. Our goal is to generalize their result in a positive direction. For this purpose consider $\mathfrak{so}(3, \mathbb{R})$ defined by the standard relations $[u, v] = w, [v, w] = u, [w, u] = v$, and define the special current Lie algebra $\mathfrak{n}_1^+$ by its basis (for instance $[u \otimes t^1, v \otimes t^1] = w \otimes t^2$, $[v \otimes t^1, w \otimes t^2] = u \otimes t^3$ and so on)

$$\begin{matrix} u \otimes t^1 \\ v \otimes t^1 \end{matrix}, w \otimes t^2, \begin{matrix} u \otimes t^3 \\ v \otimes t^3 \end{matrix}, w \otimes t^4, \begin{matrix} u \otimes t^5 \\ v \otimes t^5 \end{matrix}, w \otimes t^6, \dots$$

Theorem. *Let $\mathfrak{g} = \oplus_{i \in \mathbb{N}} \mathfrak{g}_i$ be a narrow Carnot algebra satisfying*

$$\dim \mathfrak{g}_i + \dim \mathfrak{g}_{i+1} \leq 3, \ i \in \mathbb{N},$$

that admits an integrable complex structure. Then $\mathfrak{g}$ is isomorphic to some even-dimensional quotient of $\mathfrak{n}_1^+$.